Billiard boundary perturbations and periodic orbits

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We relate the first-order quantum energy shifts due to perturbation of a billiard boundary to classical periodic orbits using the Gutzwiller relation. By using the specific example of the wedge billiard we show how the periodic orbit theory thus derived very accurately describes correlations among the energy shifts.

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I. INTRODUCTION

The study of chaotic classical systems and their quantal analogues has generated much interest in recent years [1–3]. This effort has produced a partial understanding of how classical mechanics and quantum mechanics are connected in the semiclassical regime.

In the study of chaos and the connection between classical and quantum mechanics billiard systems such as the Sinai billiard [4,5], the stadium or Bunimovich billiard [6,7], and the wedge billiard [8–11] have been illuminating paradigms. Billiard systems have some advantages over generic smooth Hamiltonian systems, such as offering a natural definition for the surface of section (often the billiard boundary) and allowing analytic maps to be derived that take the classical trajectory from one intersection of the surface of section to another. The classical mechanics of such billiard systems "scales" [12] with energy; the phase space structure scales simply as the energy changes as opposed to the complex bifurcations and changes that occur in generic Hamiltonian systems. Quantum mechanically billiards have the advantage that the energy eigenvalues and eigenfunctions are relatively easily numerically calculated. Hence the quantal and classical observables are readily compared.

A remarkable relation, due to Gutzwiller [2,13], provides a semiclassical connection bridging the classical and quantum descriptions in a particularly clear and informative way. The classical periodic orbits of the system are the crucial ingredient in the classical side of the relation whereas the quantum energy eigenvalues are the crucial ingredient of the quantum side. The Gutzwiller relation shows how to obtain the quantum energy spectrum if the periodic orbits (and their properties, such as the stability exponents, action, and period) of the classical system are known; or conversely, how to obtain information about the periodic orbit properties if the quantum energy spectrum is known. Using the Gutzwiller relation an understanding of long-range correlations in the energy spectra of chaotic systems has been achieved through an understanding of how classical periodic orbits manifest themselves in the energy spectrum.

A natural question to ask once the role of periodic orbits in the spectrum itself has been elucidated is how the periodic orbits take part in changing the spectrum as parameters of the system are varied. How do the periodic orbits manifest themselves in the first-order energy shifts?

For smooth Hamiltonian systems the connection between periodic orbits and perturbation theory via semiclassical methods along the lines of the Gutzwiller relation has been provided by several authors [14–16]. In these papers the matrix element relevant for first-order perturbation theory, namely, $\langle \psi_n | H_1 | \psi_n \rangle$, where ψ_n is the nth eigenfunction of the unperturbed system, and H_1 is the perturbation, is evaluated in a stationary phase approximation and related to periodic orbits via the Gutzwiller relation. Unfortunately for billiard systems, the smoothness of the perturbing potential H_1 is an important feature of the derivation and applicability of the formulas derived in those papers. Since billiard systems have discontinuous potentials (at the walls of the billiard), the perturbation H_1 is also discontinuous, and the results of these papers may not be used to find matrix elements and do perturbation theory on billiards.

In this paper we complete the story started by those previous authors by providing a connection between periodic orbits and first-order quantal energy shifts for billiards. We show how, using the derived formulas, we can understand correlations among the first-order energy shifts occurring in billiards. We provide a concrete example using the wedge billiard, both for pedagogic purposes and to test the accuracy of the relation we derive. We find that the theory and numerics are in very good agreement.

II. BOUNDARY PERTURBATIONS; FIRST-ORDER THEORY

We consider a billiard whose hard wall boundary is specified by some geometrical parameters we collectively denote as the vector $\boldsymbol{\eta}$. What happens if we change the shape or the position of the boundary in some small way? How do the energy eigenvalues change?

We want to change the boundary such that the change in the energy eigenvalue is small at the energy we are looking at. More precisely, this means that if the nth unperturbed energy eigenvalue is given by $E_n^{(0)}$ and the perturbed energy eigenvalue is given by E_n then the difference $E_n - E_n^{(0)} \ll 1/d_{\mathrm{TF}}(E_n^{(0)})$ where $d_{\mathrm{TF}}(E)$ is the Thomas-Fermi density of states at energy E.

Consider a continuous, smooth function $\Delta E(E)$, which is defined at the points $E=E_n^{(0)}$ by

$$\Delta E(E_n^{(0)})=E_n-E_n^{(0)} . \tag{1}$$

Consider the function

$$\sum_{n=0}^{\infty}\Delta E(E)\delta(E-E_n^{(0)}) . \tag{2}$$

Because of the appearance of the δ function, Eq. (2) has a definite meaning only when it is used as part of a kernel and subsequent integrations are carried out over E (Arfken [17] Sec. 8.7, Schiff [18] p. 55). In other words, the fundamental property of Eq. (2) is that

$$\int_0^{\infty}dE\sum_{n=0}^{\infty}\Delta E(E)\delta(E-E_n^{(0)})=\sum_{n=0}^{\infty}\Delta E(E_n^{(0)}) . \tag{3}$$

To isolate the change in energy of a particular eigenvalue, say the one at $n=n'$, integrate Eq. (2) about a small window centered on $E=E_{n'}^{(0)}$,

$$\int_{E_{n'}^{(0)}-\delta E}^{E_{n'}^{(0)}+\delta E}dE\sum_{n=0}^{\infty}\Delta E(E)\delta(E-E_n^{(0)})=\Delta E(E_{n'}^{(0)}) . \tag{4}$$

This is precisely the change in energy of the eigenvalue $E_{n'}^{(0)}$ induced by the perturbation. We would like to be able to find this function $\Delta E(E)$ to do perturbation theory. What function has the property Eq. (3), at least approximately?

We consider the quantum staircase $N(E)$, defined by

$$N(E)=\sum_{n=0}^{\infty}\Theta(E-E_n) , \tag{5}$$

as in Fig. 1. The quantum staircase counts the number of energy levels below some energy E. The solid curve in Fig. 1 shows the quantum staircase for the unperturbed billiard $N^{(0)}(E)$, the dotted curve shows the quantum staircase for the perturbed billiard $N(E)$, and the dashed curve shows the function

$$\Delta N(E)=N^{(0)}(E)-N(E) . \tag{6}$$

When used as part of a kernel in an integral over E this function $\Delta N(E)$ has very similar properties to the function $\Delta E(E)$. Analogous to Eq. (3) we have for $\Delta N(E)$

$$\int_0^{\infty}dE\sum_{n=0}^{\infty}\Delta N(E)=\sum_{n=0}^{\infty}\Delta E(E_n^{(0)}) , \tag{7}$$

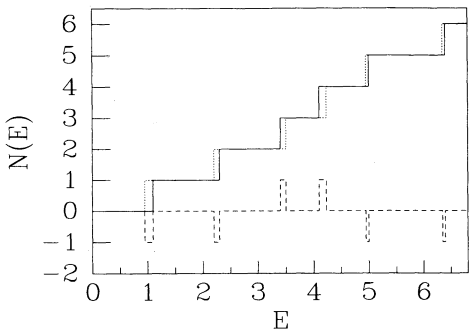

FIG. 1. A plot of the exact quantum staircase Eq. (5) for the unperturbed system (solid line), the perturbed system (dotted line), and the difference Eq. (6) (dashed line).

and analogous to Eq. (4), if one integrates about a small window centered on the eigenvalue $n=n'$ then

$$\int_{E_{n'}-\delta E}^{E_{n'}+\delta E}dE\Delta N(E)=\Delta E(E_n^{(0)}) , \tag{8}$$

provided of course that our integration window δE is "wide enough" but not "too wide." More precisely, $1/d_{\rm TF}(E_{n'})>\delta E>\Delta E(E_{n'}^{(0)})$.

The function

$$\delta_l(x)=\begin{cases}0, & x<-1/2l,\\ l, & -1/2l<x<1/2l,\\ 0, & x>1/2l,\end{cases} \tag{9}$$

approximates $\delta(x)$ for $l\gg 1$, l finite. $\Delta N(E)$ approximates $\sum_{n=0}^{\infty}\Delta E(E)\delta(E-E_n^0)$,

$$\Delta N(E)\approx\sum_{n=0}^{\infty}\Delta E(E)\delta(E-E_n^0) \tag{10}$$

in a similar sense as a series of very narrow rectangles.

III. PERIODIC ORBITS AND $\Delta N(E)$

From the Gutzwiller theory [2,10,11,13] for systems showing hard chaos $N(E,\eta)$ is given semiclassically in terms of periodic orbits by

$$\begin{aligned}N(E,\eta)&=N_{\rm TF}(E,\eta)+N_{\rm osc}(E,\eta)\\ &=N_{\rm TF}(E,\eta)+\sum_{\gamma}\sum_{k=1}^{\infty}\frac{1}{\pi k}\frac{[\sin(kS_\gamma(E,\eta)/\hbar-ku_\gamma\pi/2)+\sin(ku_\gamma\pi/2)]}{(\exp\{k\nu_\gamma/2\}-\sigma_\gamma^k\exp\{-k\nu_\gamma/2\})} ,\end{aligned} \tag{11}$$

where $N_{\rm TF}$ is the Thomas-Fermi staircase defined by

$$N_{\rm TF}(E,\eta)=\int_0^E d_{\rm TF}(E',\eta)dE' \tag{12}$$

and we have explicitly shown the dependence of the functions on the geometrical parameter η describing the boundary. In Eq. (11) γ labels the periodic orbits, k counts their multiple traversals, $S_\gamma(E,\eta)=\oint\mathbf{p}\cdot d\mathbf{q}$ is the classical action cal-

culated around one traversal of the periodic orbit, ν_γ is the stability exponent, σ_γ is the sign of the trace of the monodromy matrix, and u_γ is the Maslov index, an integer related to the number of hard wall collisions a periodic orbit undergoes and the number of times the stable and unstable manifolds rotate about the periodic orbit over one traversal.

A small change in η of $\delta\eta$ gives to first order

$$\Delta N(E,\eta)=-\frac{\partial N_{\rm TF}(E,\eta)}{\partial\eta}\delta\eta-\frac{\partial N_{\rm osc}(E,\eta)}{\partial\eta}\delta\eta$$
$$=-\frac{\partial N_{\rm TF}(E,\eta)}{\partial\eta}\delta\eta-\sum_\gamma\sum_{k=1}^{\infty}\frac{1}{\pi\hbar}\frac{[\cos(kS_\gamma(E,\eta)/\hbar-ku_\gamma\pi/2)]}{(e^{k\nu_\gamma/2}-\sigma_\gamma^k e^{-k\nu_\gamma/2})}\frac{\partial S_\gamma(E,\eta)}{\partial\eta}\delta\eta\ , \tag{13}$$

where we have kept only highest-order terms in $1/\hbar$, since we are thinking of the semiclassical regime $\hbar\to 0$.

But, because of Eq. (10), we get

$$\sum_{n=0}^{\infty}\Delta E(E,\eta)\delta(E-E_n^{(0)})\approx-\frac{\partial N_{\rm TF}(E,\eta)}{\partial\eta}\delta\eta-\sum_\gamma\sum_{k=1}^{\infty}\frac{1}{\pi\hbar}\frac{[\cos(kS_\gamma(E,\eta)/\hbar-ku_\gamma\pi/2)]}{(e^{k\nu_\gamma/2}-\sigma_\gamma^k e^{-k\nu_\gamma/2})}\frac{\partial S_\gamma(E,\eta)}{\partial\eta}\delta\eta\ . \tag{14}$$

This shows explicitly how to get the function we seek in terms of periodic orbits.

As with trying to compute individual energy levels with the Gutzwiller relation (and the periodic orbit form of the staircase function) we can quickly run into trouble trying to calculate the individual $\Delta E(E_n,\eta)$ by simply doing the sums required in Eq. (14) and integrating over a small energy range centered on $E=E_n$. This is for two reasons; the sums involved are at best conditionally convergent, and too many periodic orbits are needed to get the energy resolution we need [1,11,12]. Typically the number of periodic orbits needed grows exponentially. In the case of the wedge billiard, which we will present in Sec. IV, even for a perturbation that shifts the first energy eigenvalue by $\frac{1}{10}$ of a level spacing the classical sum on the right-hand side (RHS) of Eq. (14), as a function of E, requires over 10^6 periodic orbits to be able to resolve the δ function on the left-hand side (LHS) to $\frac{1}{10}$ of a level spacing.

A more mathematically well defined and numerically tractable method is to look for correlations in the first-order energy shifts by decomposing the energy shifts into harmonic components by doing what we call a "damped Gaussian" Fourier transform [9,16,19,20]. The damping in the transform makes the sums absolutely convergent and requires us to know only a finite number of energy levels (or conversely, periodic orbits). Also, the damping limits the resolution in the transformed function. This finite-wavelength resolution has the practical effect of smearing both the δ functions in Eq. (2) and the small rectangles in the function $\Delta N(E)$.

A damped transform of Eq. (14) is done by multiplying both sides by a function $h(E,\omega)$, [for example, $h(E,\omega)=\exp(-E^2)\exp(i\omega E)$], and integrating with respect to E to get

$$\sum_{n=0}^{\infty}\Delta E(E_n^{(0)},\eta)h(E_n^{(0)},\omega)$$
$$\approx-\int_0^\infty h(E,\omega)\frac{\partial N_{\rm TF}(E,\eta)}{\partial\eta}\delta\eta dE$$
$$-\int_0^\infty h(E,\omega)\frac{\partial N_{\rm osc}(E,\eta)}{\partial\eta}\delta\eta\, dE\ . \tag{15}$$

IV. APPLICATION TO THE WEDGE BILLIARD

We use the wedge billiard [8–11,20,21] as an example to illustrate the theory outlined in Sec. III. The wedge billiard (see Fig. 2) consists of a particle of mass m confined to the region between the y axis and the line $y=x\cot\phi$, ϕ being the wedge angle. We assume the particle makes elastic collisions with the wedge boundaries and is acted upon by a constant force mg in the negative y direction. The Hamiltonian is

$$H=\frac{1}{2m}(p_x^2+p_y^2)+mgy,\quad x\ge 0,\quad y\ge x\cot\phi\ , \tag{16}$$

where p_x and p_y are the momenta in the x and y directions, respectively.

For wedge angles less than 45° the classical system exhibits soft chaos; the phase space consists of regions of chaos interspersed with invariant tori. For $\phi=45°$ the system is integrable; the phse space is entirely filled with invariant tori. For wedge angles between 45° and 90° the classical system exhibits hard chaos [22]; there are no invariant tori in the phase space and all periodic orbits are

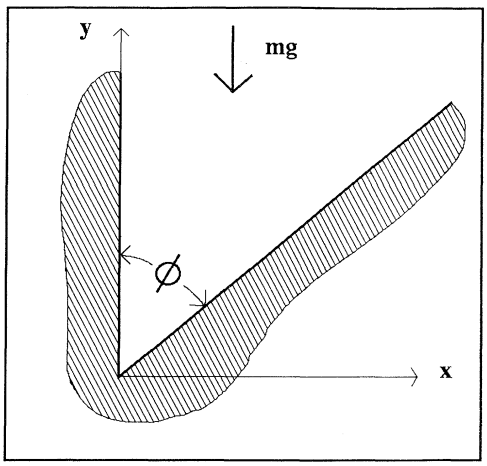

FIG. 2. The geometry of the wedge billiard.

unstable. In this paper we will be concerned with the parameter region where the wedge billiard exhibits hard chaos, $45° < \phi < 90°$.

The analogous quantum system is obtained from substituting operators into Eq. (16), giving the quantum eigenvalue equation

$$\left\{\frac{-\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]+mgy\right\}\psi_n(x,y)=E_n\psi_n(x,y)\ , \tag{17}$$

where E_n is the energy eigenvalue and $\psi_n(x,y)$ is the quantum eigenfunction that solves (17) and satisfies the boundary conditions $\psi_n(x,0)=0$ and $\psi_n(x,x\cot\phi)=0$ (i.e., it vanishes along the wedge boundaries). Recently [20,21] several thousand eigenvalues of the wedge billiard have been accurately obtained by a novel scattering approach; we refer the interested reader to the cited works for details of the numerical method.

How do the periodic orbits manifest themselves in the first-order change of the energy eigenvalues as the wedge angle ϕ is changed by $\delta\phi$? In this case the parameter we are changing and which we use to specify the boundary is ϕ; we show the explicit dependence of all functions on this parameter.

For the wedge billiard [11] we have the Thomas-Fermi staircase given by

$$N_{\rm TF}(E,\phi)=\frac{\tan\phi}{12\pi m\hbar^2 g^2}E^3 - \frac{1}{6\pi\hbar g}\left[\frac{2}{m}\right]^{1/2}(1+\sec\phi)E^{3/2}+1/6\ . \tag{18}$$

Differentiating Eq. (18) with respect to ϕ gives

$$\frac{\partial N_{\rm TF}(E,\phi)}{\partial\phi}=\frac{E^3}{12\pi m\hbar^2 g^2\cos^2\phi} - \frac{1}{6\pi\hbar g}\left[\frac{2}{m}\right]^{1/2}\frac{\sin\phi}{\cos^2\phi}E^{3/2}\ . \tag{19}$$

Because the wedge billiard scales with energy the action is a particularly simple function of energy given by $S_\gamma(E,\phi)=S_\gamma(1,\phi)E^{3/2}$.

One possible choice of $h(E,\omega)$ is the function

$$h(E,\omega)=(3\sqrt{E}\,/2)\exp[-(E^{3/2}/\alpha)^2]\cos(\omega E^{3/2})\ . \tag{20}$$

The exponential damping factor $\exp[-(E^{3/2}/\alpha)^2]$ is included to make the sums on both sides of Eq. (15) absolutely convergent [19,20]. The $3\sqrt{E}\,/2$ factor is included so that we can analytically perform the integrals required in Eq. (15). Using this $h(E,\omega)$ in Eq. (15), and switching variables to $u=E^{3/2}$ we get the Gaussian damped cosine transform

$$\sum_{n=0}^{\infty}\Delta u(u_n^{(0)},\phi)\exp[-(u_n^{(0)}/\alpha)^2]\cos(\omega u_n^{(0)}) = -\left\{a(2-\omega^2\alpha^2)\exp\left[\frac{\omega^2\alpha^2}{4}\right]+bP(\omega\alpha/2)\right\}\delta\phi - \sum_{\gamma}\sum_{k=1}^{\infty}\left\{\cos\left[k\frac{u_\gamma\pi}{2}\right]\left[P\left[[kS_\gamma(1,\phi)\hbar+\omega]\frac{\alpha}{2}\right]+P\left[[kS_\gamma(1,\phi)\hbar-\omega]\frac{\alpha}{2}\right]\right] +\sin\left[k\frac{u_\gamma\pi}{2}\right]\left[C\left[[kS_\gamma(1,\phi)\hbar+\omega]\frac{\alpha}{2}\right]+C\left[[kS_\gamma(1,\phi)\hbar-\omega]\frac{\alpha}{2}\right]\right]\right\} \times\{2\pi\hbar(e^{k\nu_\gamma/2}-\sigma_\gamma^k e^{-k\nu_\gamma/2})\}^{-1}\frac{\partial S_\gamma(1,\phi)}{\partial\phi}\delta\phi\ , \tag{21}$$

where

$$a=\frac{\alpha^3}{96m\sqrt{\pi}\hbar^2 g^2\cos^2\phi},\quad b=-\frac{1}{6\pi\hbar g}\left[\frac{2}{m}\right]^{1/2}\frac{\sin\phi}{\cos^2\phi} \tag{22}$$

$$\Delta u(u_n^{(0)},\phi)=u_n-u_n^{(0)} \tag{23}$$

and $P(x)$ and $C(x)$ are the so-called peak and crossing functions respectively defined by

$$P(x)=\frac{\alpha^2}{2}e^{-x^2}M(-1/2,1/2,x^2)\ , \tag{24}$$

$$C(x)=\frac{\alpha^2\sqrt{x}}{2}xe^{-x^2}\ , \tag{25}$$

where $M(-1/2,1/2,x^2)$ is the confluent hypergeometric function [17]. The peak and crossing functions are respectively plotted in the inset of Fig. 3. From the inset, one sees that there is a peak or a crossing as the argument x of the peak and crossing functions, respectively, is close to zero, and that the functions drop off rapidly as one moves away from $x=0$. For this reason, only the terms in Eq. (21) with argument $kS_\gamma(1,\phi)/\hbar-\omega$ contribute appreciably to the sum for $\omega>0$. Also, because both k and u_γ are integers, only one of the terms $\cos(ku_\gamma\pi/2)$ or $\sin(ku_\gamma\pi/2)$ is nonzero for a specific k in the sum. Hence, the contribution to the sum from each periodic orbit labeled by γ consists of an alternating series of peaks and crossings [9].

For a specific numerical test we choose $\phi=60°$ and $\delta\phi=-0.001°$.

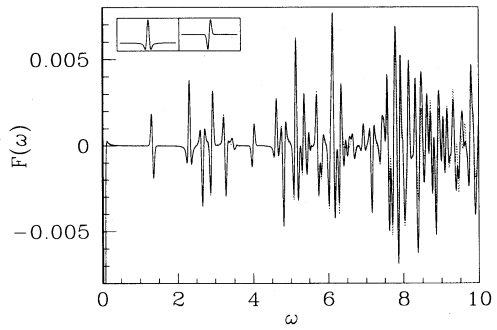

FIG. 3. $F(\omega)$ denotes the quantal LHS (solid line) and classical RHS (dotted line) of Eq. (21). All known periodic orbits with action $S_\gamma(1,\phi) \leq 10.0$ and their multiple traversals were included in the classical calculations. The differences Eq. (23) for the first 1000 quantal energy eigenvalues of the $\phi = 60.0°$ and $\phi = 59.999°$ wedges and a value of $\alpha = 50.0$ were used for the quantum calculations.

The quantum energy eigenvalues up to $n = 1000$, $E^{(0)}_{1000} = 28.23$, $u^{(0)}_{1000} = 149.99$, for the two wedge angles $\phi = 60°$ and $\phi = 59.999°$ have been calculated "exactly" quantum mechanically [21] and hence we may calculate the quantal (LHS) side of Eq. (21). The average change in the energy $d\tilde{E}_{1000}$, is found to be only (see Appendix) $|d\tilde{E}_N| = |\tilde{E}\delta\phi/\cos\phi \sin\phi| = 0.04 \ll 1$ of the average level spacing at the energy of the 1000th eigenvalue. Since the change in energy of the eigenvalues is small relative to the level spacing the theory outlined in Sec. III should be valid. We choose $\alpha = 50$, which guarantees that the damping factor $\exp[-(u^{(0)}_{1000}/\alpha)^2] < 0.0001$ and hence that most of the contribution to the quantal side of the sum comes from $n < 1000$.

On the classical, RHS of Eq. (21), the derivative $\partial S_\gamma(1,\phi)/\partial\phi$ may be numerically calculated by doing three classical calculations, one at $\phi = 60°$, another at $\phi = 59.999°$, and another at $\phi = 60.001°$. Also, the action, stability exponent, monodromy matrix, and Maslov index are calculated for the periodic orbits. As in previous work [9,11] the contributions of the vertex orbits (orbits that, at $\phi = 60°$, bounce directly into the wedge vertex) are included in the sums with a factor $\frac{1}{2}$ relative to that of nonvertex orbits.

A test of Eq. (21) is given in Fig. 3 where we have plotted the quantal LHS (solid line) versus the classical RHS (dashed line).

V. DISCUSSION

As one can see, the agreement between the two curves in Fig. 3 is qualitatively very good. They are practically indistinguishable over the range $0 \leq \omega \leq 6$, with small differences appearing for $\omega > 6$, suggesting that the theory we have derived works quite well indeed. The positions of the crossings and the peaks [which, according to Eq. (21) should occur at the actions of periodic orbits] are well reproduced, as is the amplitude of the crossings and peaks [which is related to the size of the energy shifts through the LHS of Eq. (21)]. Thus, we see that the energy shifts are intimately related to the periodic orbits of the system; the energy shifts "know" about the periodic orbits that have changed due to the perturbation and vice versa. This is exactly what was found for systems with smooth potentials [14–16].

The derivation of Eqs. (14) and (15) from Eq. (2) through the approximate association Eq. (10) is a straightforward and direct approach. Another method of deriving a semiclassical approximation to Eq. (2) may be to use the expression given in Morse and Feshbach [23], page 1061, which gives the first-order change in energy of the system in terms of integrals of unperturbed wave functions $\psi^{(0)}_n$ over the perturbed billiard surface boundary. By using the expression for the Green function in terms of the wave functions of the system, and then using the semiclassical Green function in terms of classical trajectories and evaluating the surface integrals semiclassically, one may hope to retrieve Eq. (14). Unfortunately, this author has not yet managed to achieve success using this method.

Using a scattering approach, an expression similar to (14) has been independently derived very recently [24]. Using the Sinai billiard as an example, these authors also find excellent agreement between the classical and quantal Gaussian damped transforms.

ACKNOWLEDGEMENTS

I would like to thank Christophe Rouvinez for calculating the quantum energy eigenvalues of the $\phi = 59.999°$ and $\phi = 60.0°$ wedge billiard to high accuracy. Without his work the quantal numerical calculations would not have been possible. I would also like to thank Derek Richards for useful and informative discussions, and Julie Lefebvre and David Goodings for useful and informative discussions and pointing out the section on boundary perturbations in Morse and Feshbach [23]. This work was made possible by SERC.

APPENDIX

In this Appendix we show how to calculate the mean energy shift $\overline{dE}$ of an energy eigenvalue induced by a boundary perturbation.

The total number of energy eigenvalues $N_{\rm TF}$ below some energy E is given, on average, by the Thomas-Fermi staircase $N_{\rm TF}(E,\eta)$

$$N_{\rm TF} = N_{\rm TF}(E,\eta) \ . \tag{A1}$$

Keeping the total number of eigenvalues constant, $N_{\rm TF} = \text{const}$, we vary the boundary shape and find that we must vary the eigenvalue energy as follows:

$$dN_{\rm TF} = 0 = \frac{\partial N_{\rm TF}(E,\eta)}{\partial E} dE + \frac{\partial N_{\rm TF}(E,\eta)}{\partial \eta} d\eta \tag{A2}$$

or

$$\overline{dE}(E,\eta) = -\left[\frac{\partial N_{\rm TF}(E,\eta)}{\partial \eta} d\eta\right]\left[\frac{\partial N_{\rm TF}(E,\eta)}{\partial E}\right]^{-1} . \tag{A3}$$

For the wedge billiard we have from (18) that the mean energy shift of an eigenvalue at energy E is given to dominant order in E as

$$\overline{dE}(E,\phi) = -\frac{\delta\phi}{3\cos\phi\sin\phi}E \ . \tag{A4}$$

Measured in terms of units of energy, $\tilde{E}$, where the average spacing between energy eigenvalues is 1 we have the energy shift given by

$$\overline{d\tilde{E}}(\tilde{E},\phi) = -\frac{\delta\phi}{\cos\phi\sin\phi}\tilde{E} \ . \tag{A5}$$

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